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Interval uncertainty non-cooperative games

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ABSTRACT. One of the effective tools for studying mathematical models with uncertain factors is interval analysis. In this paper, interval analysis is applied to non-cooperative games with hopeless payoff uncertainty. In other words, the model of the game assumes that the payoff of each participant can be any real number from some interval and there is no additional information about the distribution of payoffs within the interval. To determine equilibrium situations, partial ordering of intervals based on a numerical indicator is used. This allows us to reduce the original game to a new deterministic game and generalize the classical theory.

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INTRODUCTION

The theory of non-cooperative games is well developed and widely represented in the scientific and educational literature (see for example [2, 4-10, 12]). The main results are related with the definition, existence, identification, finding and selection of preferred equilibrium situations. The classical concept of equilibrium [6] assumes that the payoffs of the players in each game situation are unambiguous. If the game allows polysemy of payoffs then the concept of equilibrium changes taking into account the nature of polysemy and the degree of awareness of the players. For example, in stochastic games, the polysemy of payoffs is due to the action of random factors. The law of distribution of these factors constitutes additional information known to the players. To study equilibrium situations, the apparatus of probability theory is used.

In games with «nature», the ambiguity of payoffs is associated with the uncertainty of nature's behavior. Due to the lack of additional information, this uncertainty is characterized as «hopeless». Equilibrium is understood in terms of guaranteed payoffs. Numerous game models with hopeless uncertainty arise in applications of economic and social sciences in which the choice of criteria and objective evaluations of strategies represent a complex independent problem.

One of the effective tools for studying mathematical models with uncertain factors is interval analysis. In this section, interval analysis is applied to non-cooperative games with hopeless payoff uncertainty. In other words, the model of the game assumes that the payoff of each participant can be any real number from some interval and there is no additional information about the distribution of payoffs within the interval. To determine equilibrium situations, partial ordering of intervals based on a numerical indicator is used. This allows us to reduce the original game to a new deterministic game [10] and generalize the classical theory [5−6, 12].

1. PRELIMINARIES

By [3], we consider the space IR of regular closed real intervals $\boldsymbol{a} = [\underline{a}, \overline{a}]$, $\underline{a} \le \overline{a}$. The *center* and *radius* of the interval $\boldsymbol{a}$ will be denoted

$$a_0 = 0.5(\underline{a} + \overline{a}),\ \Delta a = 0.5(\overline{a} - \underline{a}).$$

Expressing the ends of the interval in terms of the center and the radius, we obtain an equivalent *symmetrical* representation of the interval

$$\boldsymbol{a} = [a_0 - \Delta a, a_0 + \Delta a].$$

An interval $\boldsymbol{a}$ is called *degenerate* if $\Delta a = 0$ and *non-degenerate* if $\Delta a > 0$.

Following [11], for intervals $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{IR}$ we give the definitions of inequality $\boldsymbol{a} \le \boldsymbol{b}$ in the «strong», «weak» and «central» senses:

$$\begin{gathered}\boldsymbol{a} \le \boldsymbol{b} \Leftrightarrow \big((\forall a \in \boldsymbol{a})(\forall b \in \boldsymbol{b})(a \le b)\big),\\ \boldsymbol{a} \le \boldsymbol{b} \Leftrightarrow \big((\exists a \in \boldsymbol{a})(\exists b \in \boldsymbol{b})(a \le b)\big),\\ \boldsymbol{a} \le \boldsymbol{b} \Leftrightarrow (a_0 \le b_0).\end{gathered} \tag{1.1}$$

and accept the real number

$$R(\boldsymbol{a} \le \boldsymbol{b}) = \frac{b_0 - a_0}{\Delta a + \Delta b} \tag{1.2}$$

as the *indicator* R *of the interval inequality* $\boldsymbol{a} \le \boldsymbol{b}$.

By [1, 3], for intervals $\boldsymbol{a}, \boldsymbol{b}$ from $\mathbb{IR}$ represented in symmetrical form

$$\boldsymbol{a} = [a_0 - \Delta a, a_0 + \Delta a],\ \boldsymbol{b} = [b_0 - \Delta b, b_0 + \Delta b],$$

operations of classical interval arithmetic (addition, multiplication by a real number α) are performed according to the rules

$$\boldsymbol{a} + \boldsymbol{b} = [a_0 + b_0 - \Delta a - \Delta b,\ a_0 + b_0 + \Delta a + \Delta b],$$

$$\alpha \boldsymbol{a} = [\alpha a_0 - |\alpha| \Delta a,\ \alpha a_0 + |\alpha| \Delta a].$$

Using these operations, it is easy to establish [11] the main properties of the indicator which follow from definition (1.2).

- Multiplying the interval inequality by a positive number does not change the inequality indicator; multiplying inequality by a negative number reverses the sign of the indicator.
- Inequality indicator is antisymmetric: $R(\boldsymbol{a} \le \boldsymbol{b}) = -R(\boldsymbol{b} \le \boldsymbol{a})$.
- Intervals $\boldsymbol{a}$, $\boldsymbol{b}$ with equal centers satisfy opposite inequalities $\boldsymbol{a} \le \boldsymbol{b}, \boldsymbol{b} \le \boldsymbol{a}$ with zero indicator.
- When adding interval inequalities $\boldsymbol{a} \le \boldsymbol{b}$ and $\boldsymbol{c} \le \boldsymbol{d}$ with equal indicators, an inequality $\boldsymbol{a} + \boldsymbol{c} \le \boldsymbol{b} + \boldsymbol{d}$ with the same indicator is obtained.
- If pairs of intervals $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, $\boldsymbol{d}$ have equal sums of radii then the inequality indicator $\boldsymbol{a} + \boldsymbol{c} \le \boldsymbol{b} + \boldsymbol{d}$ is equal to the arithmetic mean of the inequality indicators $\boldsymbol{a} \le \boldsymbol{b}$ and $\boldsymbol{c} \le \boldsymbol{d}$.
- For intervals $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ with pairwise equal positive sums of radii, the equality

$$R(\boldsymbol{a} \le \boldsymbol{c}) = R(\boldsymbol{a} \le \boldsymbol{b}) + R(\boldsymbol{b} \le \boldsymbol{c})$$

holds true.

According to [1, 3], we consider non-degenerate intervals $\boldsymbol{a}$, $\boldsymbol{b}$ from $\mathbb{IR}$ as sets of realization of independent uniformly distributed random variables a, b. A relation between the probability p of a random event $a \le b$ and the indicator r of the inequality $\boldsymbol{a} \le \boldsymbol{b}$ is established as follows:

$$p=\alpha(1+r)^2,\ -1\le r\le -r_1,$$
$$p=0.5(1+\beta r),\ |r|\le r_1, \quad (1.3)$$
$$p=1-\alpha(1-r)^2,\ r_1<r\le 1,$$

$$\alpha=\frac{(\Delta a+\Delta b)^2}{8\Delta a\Delta b},\ \beta=1+\frac{\Delta a}{\Delta b},\ r_1=\frac{|\Delta a-\Delta b|}{\Delta a+\Delta b}. \quad (1.4)$$

The last formulas define a continuous strictly increasing function on the segment $-1\le r\le 1$. We extend it by continuity to the entire real axis, setting

$$p=0,\ r<-1;\ p=1,\ r>1. \quad (1.5)$$

Denote the resulting function by $\pi(r)$. On the interval $-1<r<1$, the function $\pi(r)$ has a single-valued inverse function $r=\pi^{-1}(p)$.

Formulas (1.3) − (1.5) give a certain probabilistic interpretation to the relationships in which the indicator of interval inequality participates.

2. REDUCTION OF INTERVAL NON-COOPERATIVE GAMES

2.1 Model of the game

Consider the model of an interval non-cooperative game

$$\mathbf{\Gamma}_n=\langle X_1,...,X_n,\boldsymbol{H}_1,...,\boldsymbol{H}_n\rangle, \quad (2.1)$$

where X_i is the set of strategies, $\boldsymbol{H}_i : X_1\times...\times X_n\to \mathrm{IR}$ - interval payoff function of player *i*, $N=\{1,...,n\}$ – set of players. According to the rules of the game, each player $i\in N$, independently of other players, chooses a *strategy* $x_i\in X_i$. As a result, a known *situation* $x=(x_1,...,x_n)$ is formed in the game and the payoff interval $\boldsymbol{H}_i(x)$ for each player $i\in N$ is determined. The goal of the players is to «maximize» their individual payoff intervals by choosing their strategies. The task is to formalize the goals of the players and indicate the ways to find acceptable strategies.

If on the set of situations $X = X_1 \times \ldots \times X_n$ all intervals $\boldsymbol{H}_i(x) = H_i(x)$ are degenerate then the game (2.1) turns into the classical [6, 12] non-cooperative game

$$\Gamma_n = \langle X_1, \ldots, X_n, H_1, \ldots, H_n \rangle. \tag{2.2}$$

Another classic non-cooperative game

$$\Gamma_{n0} = \langle X_1, \ldots, X_n, H_{10}, \ldots, H_{n0} \rangle \tag{2.3}$$

is formed by the centers $H_{10}, \ldots, H_{n0}$ of the payoff functions. As will be seen, the game (2.3) plays an important role in characterizing the equilibrium situations of the original interval game (2.1).

2.2. Equilibrium situation

Let x be some situation in the game $\boldsymbol{\Gamma}_n$. Following [12], we denote by the symbol $x \| y_i = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ the new situation obtained from x by replacing the strategy x_i with y_i. Obviously $x \| x_i = x$ for $i \in N$.

Assume that $r = (r_1, \ldots, r_n)$ is a given vector with value in cube $[-1, 0]^n \subset \mathrm{R}^n$. We call a situation $x^{(r)}$ *acceptable* for player i if it provides a preferable payoff interval $\boldsymbol{H}_i(x^{(r)})$ with the indicator not less than $r_i \in [-1, 0]$:

$$R(\boldsymbol{H}_i(x^{(r)} \| x_i) \le \boldsymbol{H}_i(x^{(r)})) \ge r_i,\ x_i \in X_i. \tag{2.4}$$

Let us call $x^{(r)}$ an *r-equilibrium situation* if it is acceptable for all players.

Using formula (1.2), we represent inequalities (2.4) in the form

$$\begin{gathered} H_{i0}(x^{(r)}) - r_i \Delta H_i(x^{(r)}) \ge H_{i0}(x^{(r)} \| x_i) + r_i \Delta H_i(x^{(r)} \| x_i), \\ x_i \in X_i, i \in N. \end{gathered} \tag{2.5}$$

The set of *r*-equilibrium situations is denoted by X_r. For the situation $x^{(r)}$ to be *r*-equilibrium, it is sufficient that its constituent strategies $x_i^{(r)}, i \in N$ be solutions of the corresponding problems of mathematical programming

$$H_{i0}(x^{(r)} \| x_i) - r_i \Delta H_i(x^{(r)} \| x_i) \to \max,\ x_i \in X_i, i \in N. \tag{2.6}$$

Indeed, if $x_i^{(r)}$ are solutions of the problems (2.6) then

$$H_{i0}(x^{(r)}) - r_i \Delta H_i(x^{(r)}) = H_{i0}(x^{(r)} \| x_i^{(r)}) - r_i \Delta H_i(x^{(r)} \| x_i^{(r)}) \geq$$
$$\geq H_{i0}(x^{(r)} \| x_i) - r_i \Delta H_i(x^{(r)} \| x_i),\ x_i \in X_i,\ i \in N.$$

Hence, by virtue of the nonpositiveness of the product $r_i \Delta H_i(x^{(r)} \| x_i)$, it follows that

$$H_{i0}(x^{(r)}) - r_i \Delta H_i(x^{(r)}) \geq H_{i0}(x^{(r)} \| x_i) - r_i \Delta H_i(x^{(r)} \| x_i) \geq$$
$$\geq H_{i0}(x^{(r)} \| x_i) + r_i \Delta H_i(x^{(r)} \| x_i),\ x_i \in X_i, i \in N.$$

Then, by definition (2.5), the situation $x^{(r)}$ is r-equilibrium.

We note two important properties of the set of r-equilibrium situations.

1. If $X_r \neq \varnothing$ and $x^{(r)} \in X_r$ then, in accordance with formulas (1.3) - (1.5), each inequality (4.4) is satisfied with a probability not less than $\pi(r_i)$.

2. If $r, s \in [-1, 0]^n$, $r \leq s$, and $X_s \neq \varnothing$ then $X_r \neq \varnothing$ and $X_r \supset X_s$.

To check the second property, suppose $X_s \neq \varnothing$ and $x^{(s)} \in X_s$, i.e., inequalities

$$H_{i0}(x^{(s)}) - s_i \Delta H_i(x^{(s)}) \geq H_{i0}(x^{(s)} \| x_i) + s_i \Delta H_i(x^{(s)} \| x_i),$$
$$x_i \in X_i,\ i \in N,$$

are true. Then from here for $-1 \leq r_i \leq s_i \leq 0,\ \Delta H_i(x^{(s)}) \geq 0,\ i \in N$, we get

$$H_{i0}(x^{(s)}) - r_i \Delta H_i(x^{(s)}) \geq H_{i0}(x^{(s)} \| x_i) + r_i \Delta H_i(x^{(s)} \| x_i),$$
$$x_i \in X_i,\ i \in N.$$

Hence, $x^{(s)} \in X_r, X_r \neq \varnothing$ and, due to the arbitrariness of $x^{(s)} \in X_s$, the inclusion $X_r \supset X_s$ is true.

By definition (2.5), 0-equilibrium situations $x^{(0)}$ are Nash equilibrium [6] situations of the *central* game (2.3). By property 1, each situation $x^{(0)}$ satisfies the interval inequalities

$$\boldsymbol{H}_i(x^{(0)} \| x_i) \leq \boldsymbol{H}_i(x^{(0)}),\ x_i \in X_i,\ i \in N\ , \tag{2.7}$$

with probability $\geq \pi(0) = 0.5$. Moreover, the lower estimate $\pi(0)$ is the maximum of all possible ones in the sense that

$$\pi(r_i) \leq \pi(0),\ r_i \in [-1,\ 0],\ i \in N\ .$$

By property 2, a non-empty set of situations X_0 is contained in all sets $X_r, r \in [-1,0]^n$. For these two reasons, the central game Γ_{n0} is preferable to other reductions of the interval game $\boldsymbol{\Gamma}_n$ for $r \neq 0$.

We summarize. *Finding r-equilibrium situations* $x^{(r)}$ *of the interval game (2.1) for given* $r \in [-1,0]^n$ *is reduced to solving deterministic problems of mathematical programming (2.6). The central game (2.3) is preferable for the players because of the «narrowness» of the set of equilibrium situations* X_0 *and the «maximum» lower bound* $\pi(0) = 0.5$ *of the probability of fulfilling inequalities (2.7).*

It is also useful to note that the central game inherits the analytic properties of the centers and radii of the interval payoff functions (continuity, concavity, and convexity in the totality or individual groups of arguments). Therefore, the well-known [12] sufficient conditions for the existence of equilibrium strategies for the central game can be formulated in terms of the original interval game as sufficient conditions for its solvability. For example, in the original interval game, 0-equilibrium situations exist if the sets of player strategies are convex compacts of finite-dimensional spaces and the centers of interval payoff functions are continuous on the set of situations ones that are concave in the corresponding arguments on the sets of player strategies.

In conclusion of the section, we dwell on the equilibrium conditions for two important particular cases of interval games when on the set of situations X the radii of the payoff intervals are given by constant absolute errors

$$\Delta H_i(x) = \delta_i,\ i \in N\ ,$$

or permanent relative errors

$$\Delta H_i(x)/H_{i0}(x)=\alpha_i,\ i\in N\,.$$

Equilibrium conditions (2.5) in the first case take the form

$$\begin{gathered}H_{i0}(x^{(r)})\ge H_{i0}(x^{(r)}\,\|\,x_i)+2\delta_i r_i,\\ x_i\in X_i,\ i\in N,\end{gathered}\tag{2.8}$$

and in the second case –

$$\begin{gathered}(1-\alpha_i r_i)H_{i0}(x^{(r)})\ge(1+\alpha_i r_i)H_{i0}(x^{(r)}\,\|\,x_i),\\ x_i\in X_i,\ i\in N.\end{gathered}\tag{2.9}$$

2.3. Interval antagonistic games

Consider a non-cooperative game

$$\mathbf{\Gamma}_2=\langle X,Y,\boldsymbol{H},-\boldsymbol{H}\rangle$$

with interval payoff functions

$$\begin{gathered}\boldsymbol{H}(x,y)=\left[H_0(x,y)-\Delta H(x,y),\ H_0(x,y)+\Delta H(x,y\right],\\ -\boldsymbol{H}(x,y)=\left[-H_0(x,y)-\Delta H(x,y),\ -H_0(x,y)+\Delta H(x,y)\right],\end{gathered}$$

given on the set $X\times Y$ of all situations (x,y). According to the rules of the non-cooperative game, the payoffs of players 1 and 2 in the situation (x,y) are $\boldsymbol{H}(x,y)$ and $-\boldsymbol{H}(x,y)$ respectively. Consequently, an attempt by one of the players to «increase» his payoff interval will automatically lead to a «decrease» in the other player's payoff interval. Therefore, it is natural to call the game $\mathbf{\Gamma}_2$ an *antagonistic game* and designate

$$\mathbf{\Gamma}=\langle X,Y,\boldsymbol{H}\rangle.$$

Note that in contrast to the classical antagonistic game the sum of the payoffs of the players in the game $\mathbf{\Gamma}$ is, generally speaking, different from zero:

$$\boldsymbol{H}(x,y)+(-\boldsymbol{H}(x,y))=\left[-2\Delta H_0(x,y),\ 2\Delta H_0(x,y)\right],\ (x,y)\in X\times Y\,.$$

We proceed to the description of equilibrium situations. Let us choose a vector $r=(r_1,r_2)\in[-1,0]^2$. According to definition (2.5), *r*-equilibrium situations $(x^{(r)},\ y^{(r)})$ are described by the inequalities

$$H_0(x^{(r)},y^{(r)})-r_1\Delta H(x^{(r)},y^{(r)})\ge H_0(x,y^{(r)})+r_1\Delta H(x,y^{(r)}),\ x\in X,$$
$$H_0(x^{(r)},y^{(r)})+r_2\Delta H(x^{(r)},y^{(r)})\le H_0(x^{(r)},y)-r_2\Delta H(x^{(r)},y),\ y\in Y. \tag{2.10}$$

The set Z_r of *r*-equilibrium situations of the game $\mathbf{\Gamma}$ inherits the properties of the set X_r of the game $\mathbf{\Gamma}_n$. In particular, if $Z_0\ne\varnothing$ then the inclusion $Z_0\subset Z_r$ holds for any $r\in[-1,0]^2$ and $\pi(0)=0.5$ is a maximum lower estimate of the probabilities $\pi(r_1),\ \pi(r_2),\ -1\le r_1,r_2\le 0$ of fulfilling corresponding inequalities

$$\begin{aligned}\boldsymbol{H}(x,y^{(r)})&\le\boldsymbol{H}(x^{(r)},y^{(r)}),\ x\in X,\\ -\boldsymbol{H}(x^{(r)},y)&\le-\boldsymbol{H}(x^{(r)},y^{(r)}),\ y\in Y.\end{aligned} \tag{2.11}$$

By virtue of the above properties, 0-equilibrium situations more preferable for players than other *r*-equilibrium situations. Assuming $r=0$ in inequalities (2.10) after obvious transformations we convince that 0-equilibrium situations are simultaneously *saddle points* of the function $H_0(x,y)$ on the set $X\times Y$, i.e.,

$$H_0(x,y^{(0)})\le H_0(x^{(0)},y^{(0)})\le H_0(x^{(0)},y),\ x\in X,y\in Y.$$

We summarize. *For a given vector* $r\in[-1,0]^2$, *r-equilibrium situations* $(x^{(r)},y^{(r)})$ *of an antagonistic game* $\mathbf{\Gamma}=\langle X,Y,\boldsymbol{H}\rangle$ *are described by inequalities (2.10). The central antagonistic game* $\Gamma_0=\langle X,Y,H_0\rangle$ *is characterized by the least inclusion set of saddle points and the maximum lower bounds for the probabilities of fulfilling inequalities (2.11) for* $r\in[-1,0]^2$.

Example 2.1

Consider an interval antagonistic game with a payoff function

$$\boldsymbol{H}(x,y)=[(x-y)^2-\delta,\ (x-y)^2+\delta]$$

on the set of situations $(x,y)\in[0,1]^2$ with $\delta=0.2$. In terms of content, it is interpreted as a game of «evasion-approach» in conditions when the square of the distance between points x and y on the interval [0,1] is known to players with an absolute error δ.

By direct verification, we convince that there are no saddle points in the central game. We find a vector $r=(r_1,r_2)$ from the square $[-1,0]^2$ so that the set of r-equilibrium situations Z_r is not empty. According to (2.10) and (2.8), the sought for situations $(x^{(r)},\ y^{(r)})$ for any $x,y\in[0,1]$ satisfy the conditions

$$(x-y^{(r)})^2+2\delta r_1\le(x^{(r)}-y^{(r)})^2\le(x^{(r)}-y)^2-2\delta r_2 \tag{2.12}$$

and, including the most «tough» of them

$$\max_{0\le x\le 1}(x-y^{(r)})^2+2\delta r_1\le(x^{(r)}-y^{(r)})^2\le-2\delta r_2. \tag{2.13}$$

Function

$$\max_{0\le x\le 1}(x-y^{(r)})^2=\max\left\{(1-y^{(r)})^2,(y^{(r)})^2\right\}$$

on the segment $0\le y^{(r)}\le 1$ takes a minimum value of 0.25 at $y^{(r)}=0.5$. Let us use this to ensure the compatibility of inequalities (2.13). Assuming $y^{(r)}=0.5$ in (2.13), we obtain

$$0.25+2\delta r_1\le(x^{(r)}-0.5)^2\le-2\delta r_2. \tag{2.14}$$

If we accept $y^{(r)}=0.5$ as r-equilibrium strategy of the second player then the best strategy of the first player $x^{(r)}=0$ or $x^{(r)}=1$. In situations (0, 0.5), (1, 0.5), the compatibility of inequalities (2.14) depends on the numbers r_1,r_2. Choosing them from the interval $[-1,0]$ as maximum, we find

$$r_1=0,\ r_2=-0.125/\delta=-0.125/0.2=-0.6125.$$

Thus, at $r=(0,-0.6125)$ the evasion–approach game has equilibrium situations (0, 0.5) and (1, 0.5).

2.4. Interval bimatrix games

A game $\mathbf{\Gamma}_2 = \langle X_1, X_2, \boldsymbol{H}_1, \boldsymbol{H}_2 \rangle$ with two players and finite sets of strategies X_1, X_2 is called an *interval bimatrix game*. Without loss of generality, we take the sets $M = \{1,...,m\}$, $N = \{1,...,n\}$ as X_1, X_2. Compose from players' payoffs

$$\boldsymbol{H}_1(i,j) = \boldsymbol{a}_{ij},\ \boldsymbol{H}_2(i,j) = \boldsymbol{b}_{ij},\ (i,j) \in M \times N,$$

payoff matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{IR}^{m\times n}$. The goals of the players are to maximize individual payoffs by choosing pure strategies – numbers of rows and columns of matrices $\boldsymbol{A}, \boldsymbol{B}$. An interval bimatrix game $\mathbf{\Gamma}_2 = \langle M, N, \boldsymbol{H}_1, \boldsymbol{H}_2 \rangle$ in pure strategies is uniquely defined by payoff matrices so we denote it by $\mathbf{\Gamma}(\boldsymbol{A}, \boldsymbol{B})$.

2.4.1. Pure equilibrium situations

Consider an interval bimatrix game $\mathbf{\Gamma}(\boldsymbol{A}, \boldsymbol{B})$. Let us define for each situation $(i,j) \in M \times N$ the maximum number r_{ij} satisfying the inequalities

$$\begin{aligned} &R(\boldsymbol{a}_{\alpha j} \le \boldsymbol{a}_{ij}) \ge r_{ij},\ \alpha = 1,...,m,\ \alpha \ne i, \\ &R(\boldsymbol{b}_{i\beta} \le \boldsymbol{b}_{ij}) \ge r_{ij},\ \beta = 1,...,n,\ \beta \ne j. \end{aligned} \tag{2.15}$$

Denote the largest of the numbers r_{ij} by *r*. We call a situation $(i^{(r)}, j^{(r)})$ *r-equilibrium* if

$$\begin{gathered} R(\boldsymbol{a}_{ij^{(r)}} \le \boldsymbol{a}_{i^{(r)}j^{(r)}}) \ge r,\ R(\boldsymbol{b}_{i^{(r)}j} \le \boldsymbol{b}_{i^{(r)}j^{(r)}}) \ge r, \\ i = 1,...,m,\ i \ne i^{(r)}, j = 1,...,n,\ j \ne j^{(r)}. \end{gathered} \tag{2.16}$$

The characteristic property of the equilibrium situation $(i^{(r)}, j^{(r)})$ is that, according to the performance indicator of the interval inequality, the element $\boldsymbol{a}_{i^{(r)}j^{(r)}}$ is maximum in the row $i^{(r)}$ of matrix $\boldsymbol{A}$ and the element $\boldsymbol{b}_{i^{(r)}j^{(r)}}$ is maximum in the column $j^{(r)}$ of matrix $\boldsymbol{B}$.

Since the sets of strategies M and N are finite, the existence of r-equilibrium situations is beyond of doubt but the acceptability for players depends on the values of r. If $r > 1$ then the r-equilibrium situation is stable – it remains equilibrium [12] in an ordinary bimatrix game $\Gamma(A,B)$ for any $A \in \boldsymbol{A}, B \in \boldsymbol{B}$. For $|r| \leq 1$ r-equilibrium situation is the most real in the sense of the relationship between the probability and the performance indicator of the interval inequality noted in Section 1.

For each r-equilibrium situation with $r < -1$ there is another situation with a better payoff for at least one player. In this case, r-equilibrium strategies are not acceptable for the players - the game has no solution in pure strategies.

A particular case of the interval bimatrix game is the *antagonistic* matrix game $\boldsymbol{\Gamma}(\boldsymbol{A}) = \boldsymbol{\Gamma}(\boldsymbol{A}, -\boldsymbol{A})$. For an antagonistic game, condition (2.16) of the r-equilibrium situation $(i^{(r)}, j^{(r)})$ takes the form

$$
\begin{gathered}
R(\boldsymbol{a}_{i^{(r)}j^{(r)}} \leq \boldsymbol{a}_{i^{(r)}j}) \geq r,\ R(\boldsymbol{a}_{ij^{(r)}} \leq \boldsymbol{a}_{i^{(r)}j^{(r)}}) \geq r, \\
i = 1, \ldots, m,,\ i \neq i^{(r)}, j = 1, \ldots, n,\ j \neq j^{(r)}.
\end{gathered} \tag{2.17}
$$

The characteristic property of an equilibrium situation $(i^{(r)}, j^{(r)})$ in an antagonistic game $\boldsymbol{\Gamma}(\boldsymbol{A})$ is that, according to the indicator of the fulfillment of the interval inequality, the element $\boldsymbol{a}_{i^{(r)}j^{(r)}}$ is minimal in the row $i^{(r)}$ and maximal in the column $j^{(r)}$ of the matrix $\boldsymbol{A}$.

Example 2.2

Consider for $\boldsymbol{d} = [-\delta, \delta]$ the interval variant

$$
\boldsymbol{A} = \begin{pmatrix} 2+\boldsymbol{d} & \boldsymbol{d} \\ \boldsymbol{d} & 1+\boldsymbol{d} \end{pmatrix},\ \boldsymbol{B} = \begin{pmatrix} 1+\boldsymbol{d} & \boldsymbol{d} \\ \boldsymbol{d} & 2+\boldsymbol{d} \end{pmatrix}
$$

known bimatrix game «family dispute» [12] allowing the same absolute error $\delta > 0$ in the estimates of moral satisfaction of the players with the outcomes of the game. Using formulas (2.15), we find

$$r_{11} = r_{22} = 1/(2\delta),\ r_{12} = r_{21} = -1/\delta .$$

Requirement (2.16) is satisfied by two pure $1/(2\delta)$-equilibrium situations (1, 1) and (2, 2). For $\delta \in (0,\ 0.5)$ these situations are stable equilibrium, for $\delta \in [0.5,\ 1]$ - the most probable.

Example 2.3

Modified model

$$\boldsymbol{A} = \begin{pmatrix} [-1.5,-0.5] & [-11,-9] \\ 0 & [-7,-5] \end{pmatrix},\ \boldsymbol{B} = \begin{pmatrix} [-1.5,-0.5] & 0 \\ [-11,-9] & [-7,-5] \end{pmatrix}$$

of other well-known bimatrix game «prisoner's dilemma» [12] takes into account the ambiguity of the term of punishment for a crime. Here the only pure 2-equilibrium situation (2, 2) corresponds to the equilibrium condition (2.16).

Example 2.4

Consider the interval extension

$$\boldsymbol{A} = \begin{pmatrix} -100\boldsymbol{a} & 50\boldsymbol{a} \\ 100\boldsymbol{a} & -100\boldsymbol{a} \end{pmatrix},\ \boldsymbol{a} = [1-\alpha, 1+\alpha],$$

of the antagonistic matrix game «the pursuit of Sherlock Holmes» [5]. We consider the parameter $\alpha \in (0,1)$ to be the relative error of the players' payoffs. Using formulas (4.17), we establish that all situations in the game are pure r-equilibrium for $r = -1/\alpha < -1$. In this case, the game has no solution in pure strategies.

2.4.2. Mixed equilibrium situations

Consider an interval bimatrix game $\mathbf{\Gamma}(\boldsymbol{A},\boldsymbol{B})$. Interval non-cooperative game $\mathbf{\Gamma}_2(X,Y,\boldsymbol{H}_1,\boldsymbol{H}_2)$ with the sets of player strategies

$$X=\left\{x\in \mathrm{R}^m : x\geq 0,\, a'x=1\right\},\, Y=\left\{y\in \mathrm{R}^n : y\geq 0,\, b'y=1\right\}$$

and payoff functions

$$\boldsymbol{H}_1(x,y)=x'\boldsymbol{A}y,\ \ \boldsymbol{H}_2(x,y)=x'\boldsymbol{B}y$$

will be called the *mixed extension* of the game $\mathbf{\Gamma}(\boldsymbol{A},\boldsymbol{B})$ and denoted by $\mathbf{\Gamma}\left\langle \boldsymbol{A},\boldsymbol{B}\right\rangle$. In the above formulas are used column vectors, the prime is the sign of transposition, $a\in \mathrm{R}^m,\, b\in \mathrm{R}^n$ are vectors with coordinates equal to 1, $a'x$ is the scalar product of vectors a,x.

The vectors $x\in X, y\in Y$ are called the *mixed strategies* of the players. They have the same probabilistic interpretation as in the usual bimatrix game [5], i.e. the coordinates of the mixed strategies are the probabilities of choosing the corresponding rows and columns of the payoff matrices.

Let us pass to the study of equilibrium situations of the game $\mathbf{\Gamma}\left\langle \boldsymbol{A},\boldsymbol{B}\right\rangle$. Taking into account the non-negativity of mixed strategies, we express the centers and radii of the payoff functions in terms of the centers and radii of the payoff matrices. In the accepted notation, we get

$$\begin{aligned} H_{10}(x,y)&=x'A_0y,\ \Delta H_1(x,y)=x'\Delta Ay,\\ H_{20}(x,y)&=x'B_0y,\ \Delta H_2(x,y)=x'\Delta By. \end{aligned}$$

We choose a vector $r=(r_1,r_2)\in[-1,0]^2$. Definition (2.16) of an *r*-equilibrium situation $(x^{(r)},\, y^{(r)})$ takes the form

$$(x^{(r)})'(A_0-r_1\Delta A)y^{(r)}\geq x'(A_0+r_1\Delta A)y^{(r)},\ x\in X, \tag{2.18}$$

$$(x^{(r)})'(B_0-r_2\Delta B)y^{(r)}\geq (x^{(r)})'(B_0+r_2\Delta B)y,\ y\in Y. \tag{2.19}$$

Connect with conditions (2.18), (2.19) the systems of linear inequalities and equalities

$$(A_0 - r_1\Delta A)y \geq v_1 a \geq (A_0 + r_1\Delta A)y,\ y \in Y, \tag{2.20}$$

$$(B_0 - r_2\Delta B)'x \geq v_2 b \geq (B_0 + r_2\Delta B)'x,\ x \in X, \tag{2.21}$$

with unknown x, y, v_1, v_2. The relation of systems of inequalities with the solution of a bimatrix game is given in the following statements.

Lemma 2.1. *If for a given vector* $r \in [-1,0]^2$ *there exists a pair* $(x^{(r)}, v_2), (y^{(r)}, v_1)$ *that satisfies conditions (2.20), (2.21) then* $(x^{(r)} y^{(r)})$ *is an r-equilibrium situation in the game* $\mathbf{\Gamma}\langle \boldsymbol{A}, \boldsymbol{B}\rangle$.

Indeed, by the conditions of Lemma 2.1, the vectors $x^{(r)}, y^{(r)}$ are strategies and the pair $(y^{(r)}, v_1)$ satisfies inequalities (2.20) in the form

$$(A_0 - r_1\Delta A)y^{(r)} \geq v_1 a,\ v_1 a \geq (A_0 + r_1\Delta A)y^{(r)}.$$

Multiplying the first inequality by $(x^{(r)})'$ on the left and the second inequality on the left by an arbitrary strategy x'. Taking into account the properties of the strategies, we obtain the relations

$$(x^{(r)})'(A_0 - r_1\Delta A)y^{(r)} \geq v_1,\ v_1 \geq x'(A_0 + r_1\Delta A)y^{(r)},$$

equivalent to condition (2.18). Similarly, using inequality (2.21), we verify that the vector $x^{(r)}$ satisfies condition (2.19). The assertion has been proven.

Lemma 2.2. *If* $r \in [-1,0]^2$ *then for the compatibility of conditions (2.20), (2.21) it is sufficient that the systems of linear equations*

$$A_0 y - v_1 a = 0;\ B_0' x - v_2 b = 0 \tag{2.22}$$

have solutions $x \in X, y \in Y$.

Lemma 2.3. *Solutions*

$$y \geq 0,\ v_1 = x'A_0 y;\ x \geq 0,\ v_2 = x'B_0 y$$

of the systems of equations (2.22) form an equilibrium situation (x, y) *in the central bimatrix game.*

The proof of Lemmas 2.2, 2.3 is not difficult. Note that the numbers v_1, v_2 in Lemma 2.3 have the meaning of the expected payoffs of the players.

Example 2.5

Based on the game «pursuit of Sherlock Holmes» from Example 2.4, we will construct a bimatrix game $\mathbf{\Gamma}\langle \boldsymbol{A}, -\boldsymbol{A}\rangle$. Applying Lemma 2.3 to it, we find the mixed equilibrium

$$x = (4/7, 3/7),\ y = (3/7, 4/7),\ v_1 = -100/7,\ v_2 = 100/7,$$

which coincides with that given in the monograph [5].

The above results and examples show that the concept of interval inequality indicator can be useful in studying games with interval payoff uncertainty. Thanks to the indicator, it becomes possible to introduce a partial order relation in the set of payoffs and to determine the equilibrium situations of the game. The equilibrium conditions differ from the previously known ones by less rigidity and greater parameterization, which leads to an expansion of the set of equilibrium situations. As a result, players have additional opportunities to find compromise equilibrium situations.

A rational choice of the vector of indicators also contributes to a compromise in the game. Each of its components determines the probability with which the corresponding player finds his equilibrium strategy. Obviously, all players are equally interested in the formation of a «reliable» equilibrium situation, so it is beneficial for them to choose all the coordinates of the vector of indicators equal. Then the set of equilibrium situations and preferred payoff intervals depend on one parameter which simplifies the search for a compromise equilibrium. The central game serves as an example of the logical completion of these conclusions if it has a non-empty set of situations.

CONCLUSION

It is well-known that interval analysis is one of the effective tools for studying mathematical models with uncertain factors. In this manuscript, we applied interval analysis to non-cooperative games with hopeless payoff uncertainty, that is, to the model of the game in which it is assumed that the payoff of each participant can be any real number from some interval and there is no any additional information about the distribution of payoffs within the interval. To determine equilibrium situations, partial ordering of intervals based on interval indicator is used. Obtained results (Lemmas 2.1-2.3) allowed us to transform the original game to a new deterministic game and generalize the classical theory.

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